SOLVING THE K(2,2) EQUATION BY MEANS OF THE Q-HOMOTOPY ANALYSIS METHOD (Q-HAM)

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ABSTRACT

By means of the q-homotopy analysis method (q-HAM), the solution of the K(2,2) equation was obtained in this paper. Comparison of q- HAM with the Homotopy analysis method (HAM) and the Homotopy perturbation method (HPM) are made, The results reveal that the q-HAM has more accuracy than the others.

Keywords: *q-Homotopy Analysis Method (q-HAM), K(2,2) equation.*

1. INTRODUCTION

Nonlinear partial differential equations are known to describe a wide variety of phenomena not only in physics, where applications extend over magneto fluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasma, just to name a few, but also in biology and chemistry, and several other fields.

Several methods have been suggested to solve nonlinear equations. These methods include the Homotopy perturbation method (HPM) [11], Luapanov's artificial small parameter method[21], Adomian decomposition method [2,25], variation iterative method [22,28] and so on. Homotopy analysis method (HAM), first proposed by Liao in his Ph.D dissertation[18], is an elegant method which has proved its effectiveness and efficiency in solving many types of nonlinear equations [1,4,5,8-10,23,26,27]. The HAM contains a certain auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution [20]. In 2005 Liao [19] has pointed out that the HPM is only a special case of the HAM (The case of $h = -1$). El-Tawil and Huseen [6] proposed a method namely q-homotopy analysis method (q-HAM) which is more general method of homotopy analysis method (HAM) , The q-HAM contains an auxiliary parameter n as well as h such that the cases of (q-HAM ; $n = 1$) the standard homotopy analysis method (HAM) can be reached. The q-HAM has been successfully applied to solve many types of nonlinear problems [6, 7, 12-17]. Rosenan and Hyman [24] reported a class of partial differential equations

$$u_t + (u^m)_x + (u^n)_{xxx} = 0 \;\; , m > 0, 1 < n \leq 3,$$

which is a generalization of the the Korteweg-deVries (KdV) equation.These equations with the values of m and n are denoted by $K(m,n)$. The aim of the present work is to effectively employ the q-HAM to establish the solutions for one of these partial differential equations; namely, K(2,2) equation which is given by

$$u_t + (u^2)_x + (u^2)_{xxx} = 0$$

This equation plays an important role in the research of motion laws of liquid drop and mixed flowing matter. Comparison of the present method with the HAM and HPM is also presented in this paper.

2. BASIC IDEA OF Q-HOMOTOPY ANALYSIS METHOD (Q-HAM)

Consider the following differential equation

$$N[u(x,t)] - f(x,t) = 0 \qquad (1)$$

where N is a nonlinear operator, (x,t) denotes independent variables, $f(x,t)$ is a known function and $u(x,t)$ is an unknown function.

Let us construct the so-called zero-order deformation equation

$$(1-nq)L[\emptyset(x,t;q) - u_0(x,t)] = qhH(x,t)(N[\emptyset(x,t;q)] - f(x,t)), \qquad (2)$$

where $n \geq 1$, $q \in [0,\frac{1}{n}]$ denotes the so-called embedded parameter , L is an auxiliary linear operator with the property $L[f] = 0 \; when \; f = 0$, $h \neq 0$ is an auxiliary parameter, $H(x,t)$ denotes a non-zero auxiliary function. It is obvious that when $q = 0 \; and \; q = \frac{1}{n}$ equation (2) becomes:

$$\emptyset(x,t;0) = u_0(x,t) , \qquad \emptyset\left(x,t;\frac{1}{n}\right) = u(x,t) \qquad (3)$$

Respectively. Thus as q increases from 0 to $\frac{1}{n}$, the solution $\emptyset(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Having the freedom to choose $u_0(x,t), L, h, H(x,t)$, we can assume that all of them can be properly chosen so that the solution $\emptyset(x,t;q)$ of equation (2) exists for $q \in [0,\frac{1}{n}]$.

Expanding $\emptyset(x,t;q)$ in Taylor series, one has:

$$\emptyset(x,t;q) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t) q^m , \qquad (4)$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \emptyset(x,t;q)}{\partial q^m}\Big|_{q=0} \qquad (5)$$

Assume that $h, H(x,t), u_0(x,t), L$ are so properly chosen such that the series (4) converges at $q=\frac{1}{n}$ and

$$u(x,t) = \emptyset\left(x,t;\frac{1}{n}\right) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)\left(\frac{1}{n}\right)^m \qquad (6)$$

Defining the vector $u_r(x,t) = \{u_0(x,t), u_1(x,t), u_2(x,t), \dots, u_r(x,t)\}$.
Differentiating equation (2) m times with respect to q and then setting $q=0$ and finally dividing them by $m!$ we have the so-called m^{th} order deformation equation

$$L[u_m(x,t) - k_m u_{m-1}(x,t)] = hH(x,t)R_m(\vec{u}_{m-1}(x,t)), \qquad (7)$$

where

$$R_m(\vec{u}_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}(N[\emptyset(x,t;q)] - f(x,t))}{\partial q^{m-1}}\Big|_{q=0} \qquad (8)$$

and

$$k_m = \begin{cases} 0 & m \le 1 \\ n & otherwise \end{cases} \qquad (9)$$

It should be emphasized that $u_m(x,t)$ for $m \ge 1$ is governed by the linear equation (7) with linear boundary conditions that come from the original problem. Due to the existence of the factor $\left(\frac{1}{n}\right)^m$, more chances for convergence may occur or even much faster convergence can be obtained better than the standard HAM. It should be noted that the cases of $(n=1)$ in equation (2), standard HAM can be reached.

3. APPLICATIONS

Consider the following K(2,2) equation [3]

$$u_t + (u^2)_x + (u^2)_{xxx} = 0\,,\; u(x,0) = x \qquad (10)$$

The exact solution of this problem is

$$u(x,t) = \frac{x}{1+2t} \qquad (11)$$

This problem solved by HAM [3]. For q- HAM solution we choose the linear operator

$$L[\emptyset(x,t;q)] = \frac{\partial\emptyset(x,t;q)}{\partial t} \tag{12}$$

with the property $L[c_1] = 0$, where c_1 is constant.

Using initial approximation $u_0(x,t) = x$, we define a nonlinear operator as

$$N[\emptyset(x,t;q)] = \frac{\partial\emptyset(x,t;q)}{\partial t} + \frac{\partial(\emptyset^2(x,t;q))}{\partial x} + \frac{\partial^3(\emptyset^2(x,t;q))}{\partial x^3}$$

We construct the zero order deformation equation

$$(1-nq)L[\emptyset(x,t;q) - u_0(x,t)] = qhH(x,t)N[\emptyset(x,t;q)].$$

We can take $H(x,t) = 1$, and the m^{th} order deformation equation is

$$L[u_m(x,t) - k_m u_{m-1}(x,t)] = hR_m(\bar{u}_{m-1}(x,t)) \tag{13}$$

with the initial conditions for $m \geq 1$

$$u_m(x,0) = 0\,, \tag{14}$$

where k_m as define by (9) and

$$R_m(\bar{u}_{m-1}(x,t)) = \frac{\partial u_{m-1}(x,t)}{\partial t} + \frac{\partial}{\partial x}\sum_{i=1}^{m-1} u_i(x,t)u_{m-1-i}(x,t) +$$

$$\frac{\partial^3}{x^3}\left(\sum_{i=1}^{m-1} u_i(x,t)u_{m-1-i}(x,t)\right)$$

Now the solution of equation (10) for $m \geq 1$ becomes

$$u_m(x,t) = k_m u_{m-1}(x,t) + h\int R_m(\bar{u}_{m-1}(x,s))ds + c_1,$$

where the constant of integration c_1 is determined by the initial conditions (14). Then, the components of the solution using q- HAM are

$$u_m(x,t) = 2hxt\,(2ht + h + n)^{m-1} \text{ for } m = 1,2,3,\ldots$$

As special case if $n = 1$ and $h = -1$, then we obtain the same result in [3].

Now the series solution expression by q- HAM can be written in the form

$$u(x,;n;h) \cong U_M(x,t;n;h) = \sum_{i=0}^{M} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i \tag{15}$$

Equation (15) is an approximate solution to the problem (10) in terms of the convergence parameters h and n . To find the valid region of h , the h-curves given by the 10th order q-HAM approximation at different values of x, t and n are drawn in figures $(1-7)$. These figures show the interval of h at which the value of $U_{10}(x,t;n)$ is constant at certain values of x, t and n. We choose the horizontal line parallel to $x-axis$ (h) as a valid region which provides us with a simple way to adjust and control the convergence region of the series solution (16). From these figures, the valid intersection region of h for the values of x, t and n in the curves becomes larger as n increase. Figures $(8-10)$ show the comparison between U_5, U_7 and U_{10} using different values of n with the exact solution (11). Figure (11) shows the comparison between U_{10} of HAM, U_{10} of HPM and U_{10} of q-HAM using different values of n with the exact solution (11), which indicates that the speed of convergence for q-HAM with $n > 1$ is faster than $n = 1$ (HAM) and $(n = 1; h = -1)$ (HPM). Figure (12) shows the HPM solution, is different from the exact solution given in Figure (15), Figure (13) shows the HAM solution with $0 \leq t \leq 4.5$. However, when we increase slightly the range of t to $0 \leq t \leq 8.5$, the shape of the HAM solution, as shown in Figure (14), is different from the exact solution given in Figure (15). On the other hand, the q-HAM $(n = 100)$ solution has the same shape as the exact solution even for larger range of t, i.e. $0 \leq t \leq 8.5$ as shown in Figure (16). Table (1) shows the comparison between the 10^{th}order approximations of HAM, HPM (HAM; $h = -1$) and q-HAM at different values of n with the exact solution of (10). Therefore, based on these present results, we can say that q-HAM is more effective than HAM and HPM.

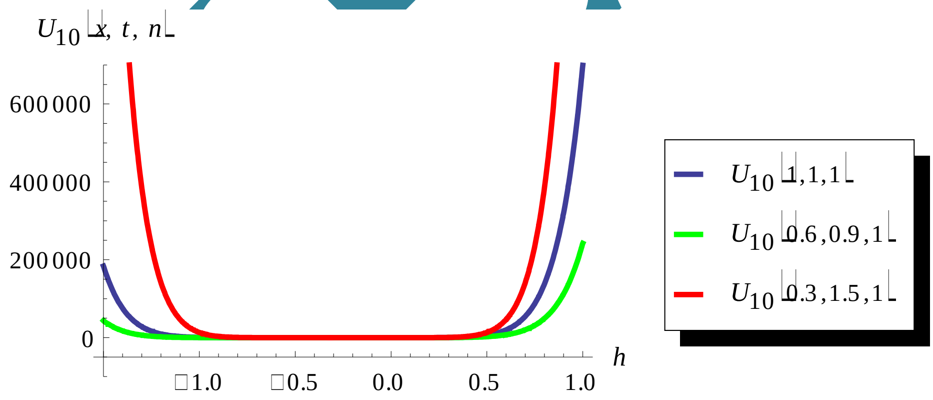


Figure (1) : h - curve for the HAM (q-HAM; $n = 1$) approximation solution $U_{10}(x,t;1)$ of problem (10) at different values of x and t.

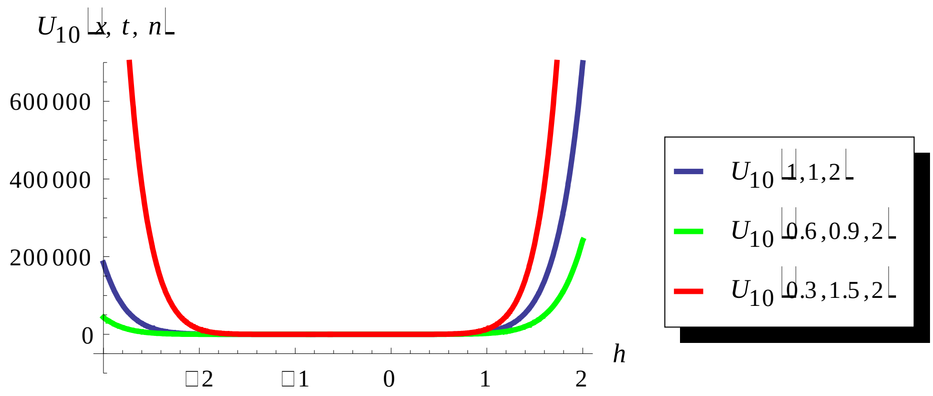


Figure (2) : h - curve for the (q-HAM; $n = 2$) approximation solution $U_{10}(x, t; 2)$ of problem (10) at different values of x and t.

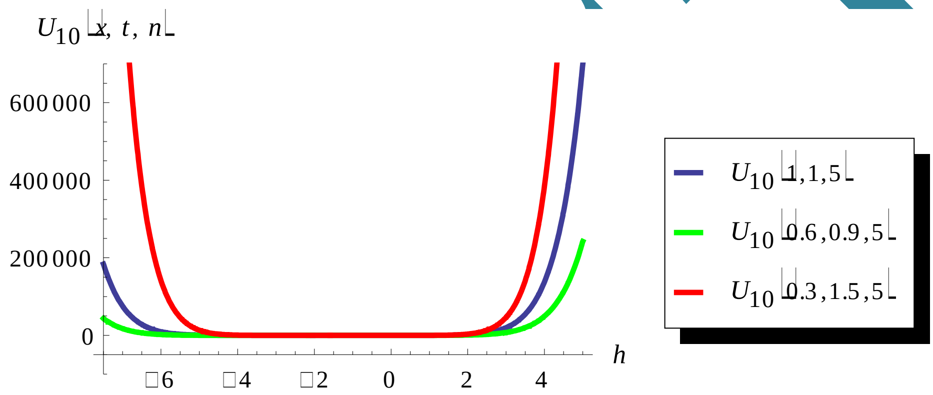


Figure (3) : h - curve for the (q-HAM; $n = 5$) approximation solution $U_{10}(x, t; 5)$ of problem (10) at different values of x and t.

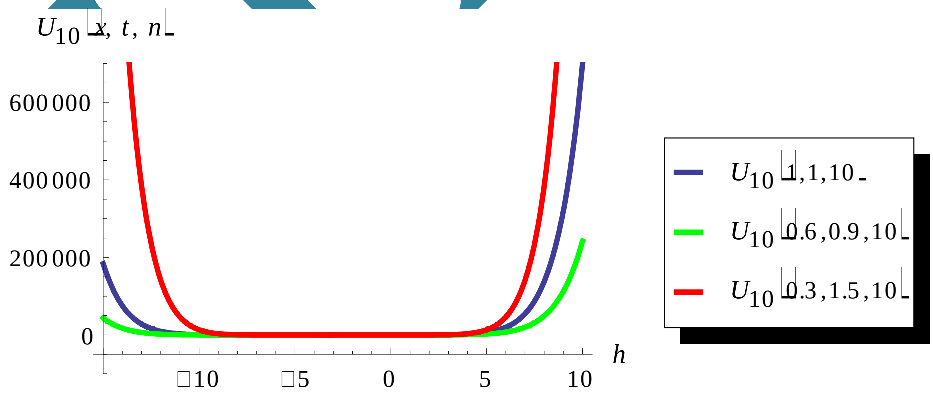


Figure (4) : h - curve for the (q-HAM; $n = 10$) approximation solution $U_{10}(x, t; 10)$ of problem (10) at different values of x and t.

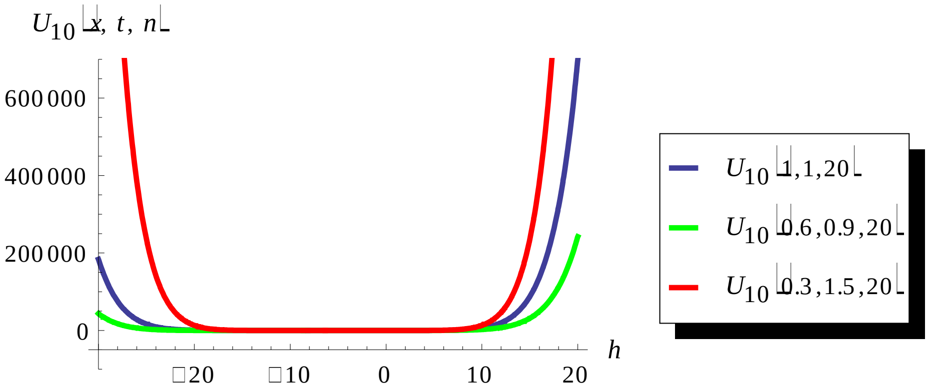


Figure (5) : h - curve for the (q-HAM; $n = 20$) approximation solution $U_{10}(x, t; 20)$ of problem (10) at different values of x and t.

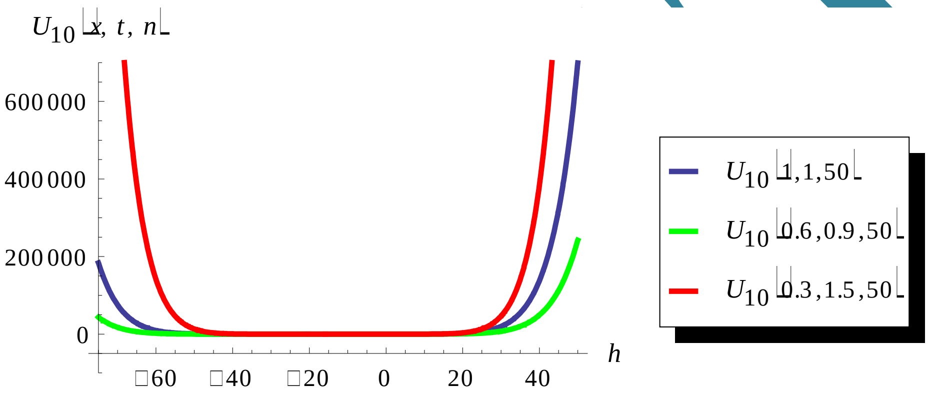


Figure (6) : h - curve for the (q-HAM; $n = 50$) approximation solution $U_{10}(x, t; 50)$ of problem (10) at different values of x and t.

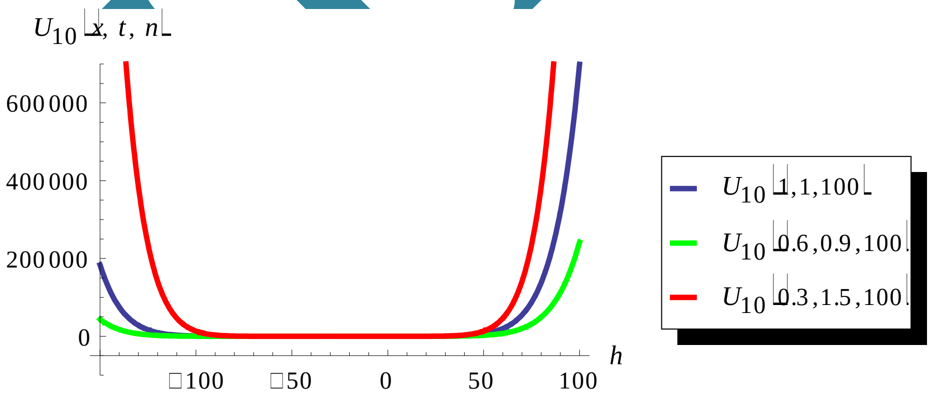


Figure (7) : h - curve for the (q-HAM; $n = 100$) approximation solution $U_{10}(x, t; 100)$ of problem (10) at different values of x and t.

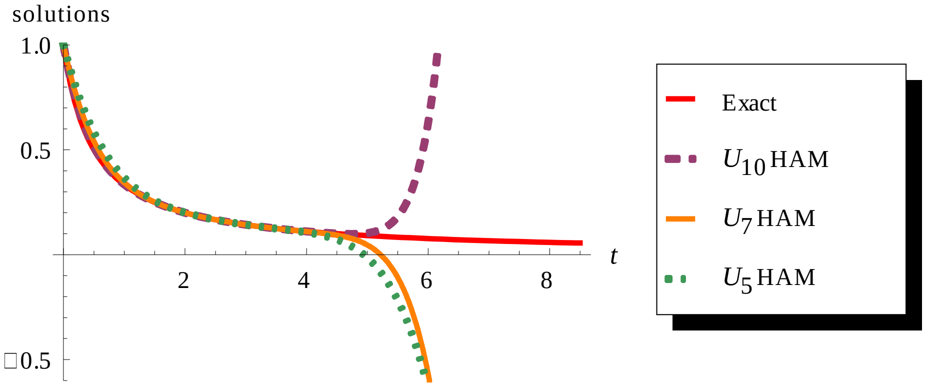


Figure (8): Comparison between U_5, U_7, U_{10} of HAM (q-HAM; $n = 1$) and exact solution of (10) at $x = 1$ with $h = -0.15$, $0 < t \leq 8.5$

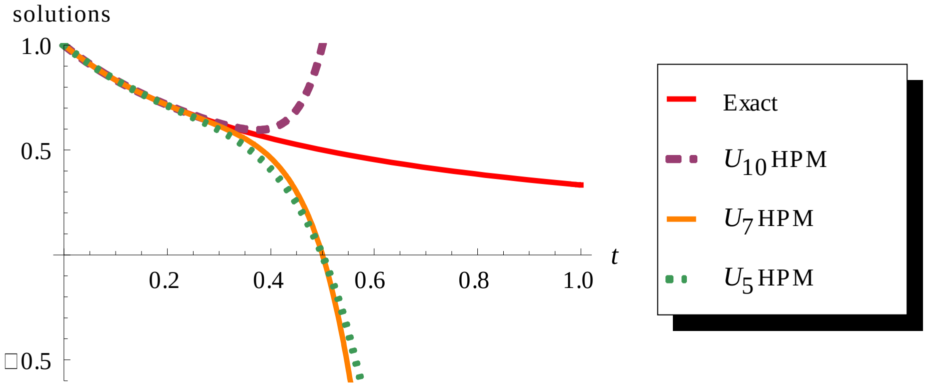
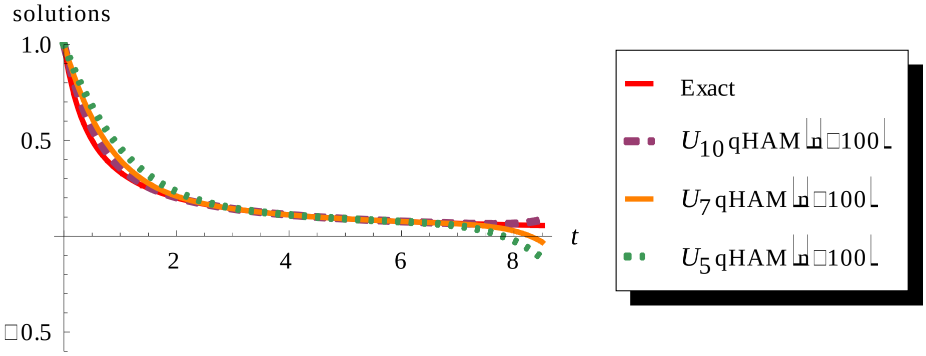


Figure (9): Comparison between U_5, U_7, U_{10} of HPM (HAM; $h = -1$) and exact solution of (10) at $x = 1$ with $h = -0.15$, $0 < t \leq 1$



Figure(10): Comparison between U_5, U_7, U_{10} of (q-HAM ; $n = 100$) and exact solution of (10) at $x = 1$ with $h = -9.5$, $0 < t \leq 8.5$

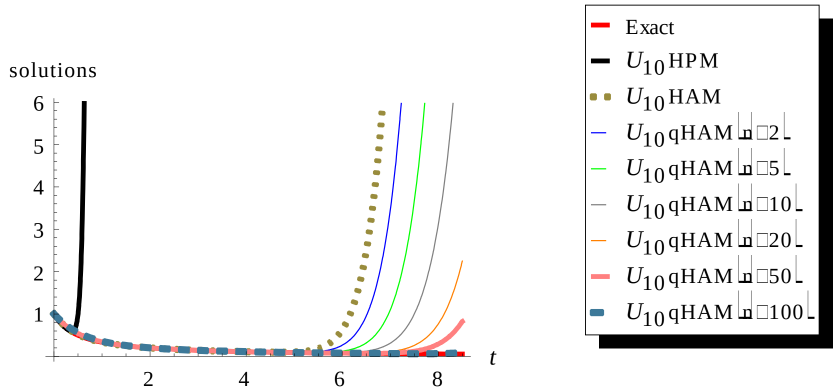


Figure (11): Comparison between U_{10} of HAM (q-HAM;($n = 1$)) , U_{10} of HPM (HAM; $h = -1$) and (q-HAM ; ($n = 2, 5, 10, 20, 50, 100$) with exact solution of (10) at $x = 1$ with ($h = -0.15, -0.285, -0.67, -1.25, -2.32, -5.5, -9.5$), $respectively$, $0 \le t \le 8.5$.

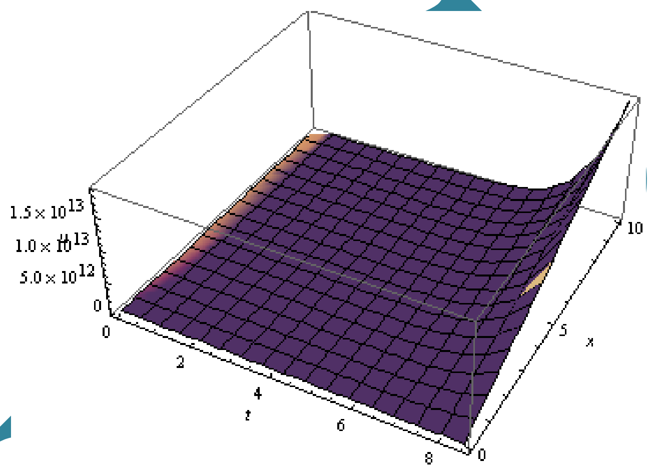


Figure (12): The 10th order solution HPM (HAM ; $h = -1$) approximate for problem (10) at $0 \le x \le 10$; $0 \le t \le 8.5$.

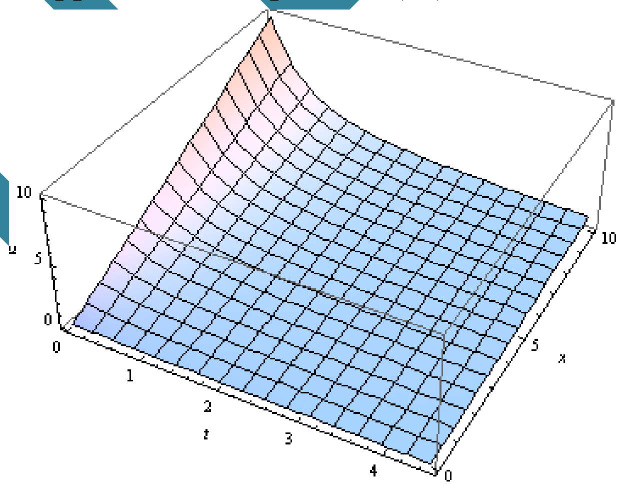


Figure (13): The 10th order solution HAM (q-HAM ; $n = 1$) approximate for problem (10) at $0 \le x \le 10$; $0 \le t \le 4.5$.

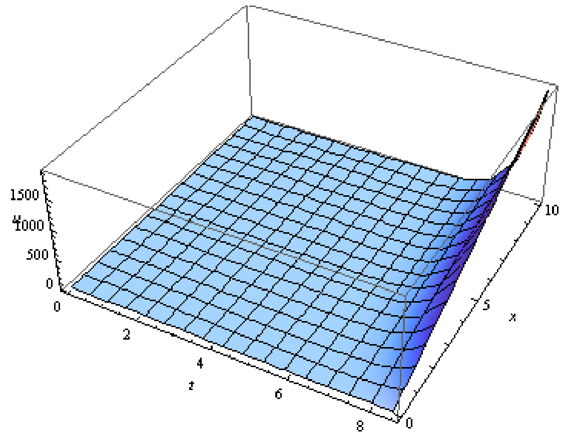


Figure (14): The 10th order solution HAM (q-HAM ; $n = 1$) approximate for problem (10) at $0 \leq x \leq 10$; $0 \leq t \leq 8.5$.

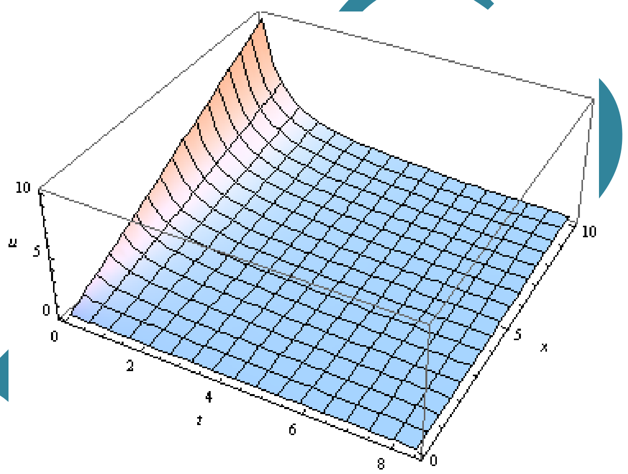


Figure (15): The exact solution for problem (10) at $0 \leq x \leq 10$; $0 \leq t \leq 8.5$.

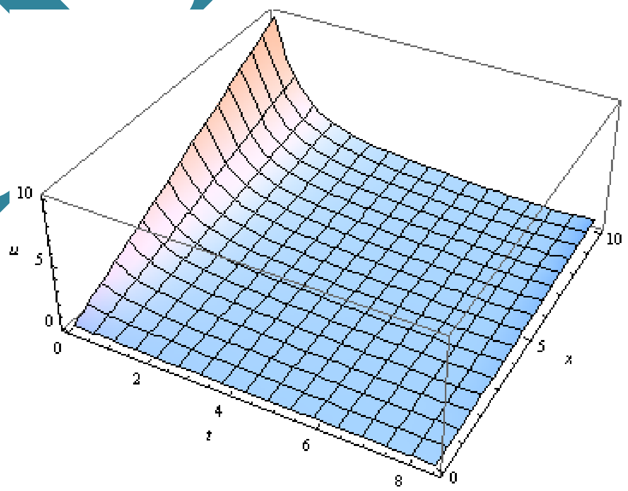


Figure (16): The 10th order solution q-HAM ($n = 100$) approximate for problem (10) at $0 \leq x \leq 10$; $0 \leq t \leq 8.5$.

x	t	u_{10} HPM	u_{10} HAM	u_{10} q-HAM (n=2)	u_{10} q-HAM (n=5)	u_{10} q-HAM (n=10)	u_{10} q-HAM (n=20)	u_{10} q-HAM (n=50)	u_{10} q-HAM (n=100)	Exact solution
0. 5	0 1 2 3 4 5 6 7	0.5 341.5 419431. 2.59141 e7 4.77219 e8 4.54545 e9 2.85772 e10 1.34986 e11	0.5 0.1675 11 0.1 0.0714 286 0.0555 678 0.0515 74 0.3148 02 4.3795 1	0.5 0.1679 27 0.1000 02 0.0714 286 0.0555 57 0.0470 294 0.1320 35 1.7258	0.5 0.1686 16 0.1000 06 0.0714 286 0.0555 556 0.0457 148 0.0618 094 0.5488 24	0.5 0.1696 98 0.1000 22 0.0714 286 0.0555 556 0.0454 795 0.0426 592 0.1561 02	0.5 0.1712 94 0.1000 68 0.0714 286 0.0555 556 0.0454 557 0.0389 898 0.0563 12	0.5 0.1727 43 0.1001 36 0.0714 287 0.0555 556 0.0454 546 0.0385 613 0.0396 159	0.5 0.1783 06 0.1006 36 0.0714 362 0.0555 556 0.0454 545 0.0384 618 0.0334 231	0.5 0.1666 67 0.1 0.0714 286 0.0555 556 0.0454 545 0.0384 615 0.0333 333
2. 5	0 1 2 3 4 5 6 7	2.5 1707.5 2.09715 e6 1.2957e 8 2.38609 e9 2.27273 e10 1.42886 e11 6.74928 e11	2.5 0.8375 55 0.5000 02 0.3571 43 0.2778 39 0.2578 7 1.5740 1 21.897 5	2.5 0.8396 37 0.5000 08 0.3571 43 0.2777 85 0.2351 47 0.6601 74 8.6290 1	2.5 0.8430 8 0.5000 31 0.3571 43 0.2777 78 0.2285 74 0.3090 47 2.7441 2	2.5 0.8484 92 0.5001 1 0.3571 43 0.2777 78 0.2273 98 0.2132 96 0.7805 1	2.5 0.8564 71 0.5003 42 0.3571 43 0.2777 78 0.2272 79 0.1949 49 0.2815 6	2.5 0.8637 14 0.5006 81 0.3571 44 0.2777 78 0.2272 73 0.1928 06 0.1980 8	2.5 0.8915 31 0.5031 81 0.3571 81 0.2777 78 0.2272 73 0.1923 09 0.1671 15	2.5 0.8333 33 0.5 0.3571 43 0.2777 78 0.2272 73 0.1923 08 0.1666 67
5	0 1 2 3 4	5 3415 4.19431 e6 2.59141	5 1.6751 1 1 0.7142	5 1.6792 7 1.0000 2	5 1.6861 6 1.0000 6	5 1.6969 8 1.0002 2	5 1.7129 4 1.0006 8	5 1.7274 3 1.0013 6	5 1.7830 6 1.0063 6	5 1.6666 7 1 0.7142

	5 6 7	e8 4.77219 e9 4.54545 e10 2.85772 e11 1.34986 e12	86 0.5556 78 0.5157 4 3.1480 2 43.795 1	0.7142 86 0.5555 7 0.4702 94 1.3203 5 17.258	0.7142 86 0.5555 56 0.4571 48 0.6180 94 5.4882 4	0.7142 86 0.5555 56 0.4547 95 0.4265 92 1.5610 2	0.7142 86 0.5555 56 0.4545 57 0.3898 98 0.5631 2	0.7142 87 0.5555 56 0.4545 46 0.3856 13 0.3961 59	0.7143 62 0.5555 56 0.4545 45 0.3846 18 0.3342 31	86 0.5555 56 0.4545 45 0.3846 15 0.3333 33
7. 5	0 1 2 3 4 5 6 7	7.5 5122.5 6.29146 e6 3.88711 e8 7.15828 e9 6.81818 e10 4.28659 e11 2.02478 e12	7.5 2.5126 6 1.5000 1 1.0714 3 0.8335 17 0.7736 1 4.7220 2 65.692 6	7.5 2.5189 1 1.5000 2 1.0714 3 0.8333 55 0.7054 41 1.9805 2 25.887	7.5 2.5292 4 1.5000 9 1.0714 3 0.8333 34 0.6857 22 0.9271 41 8.2323 5	7.5 2.5454 7 1.5003 3 1.0714 3 0.8333 33 0.6821 93 0.6398 88 2.3415 3	7.5 2.5694 1 1.5010 2 1.0714 3 0.8333 33 0.6818 36 0.5848 47 0.8446 79	7.5 2.5911 4 1.5020 4 1.0714 3 0.8333 33 0.6818 19 0.5784 19 0.5942 39	7.5 2.6745 9 1.5095 4 1.0715 4 0.8333 33 0.6818 18 0.5769 27 0.5013 46	7.5 2.5 1.5 1.0714 3 0.8333 33 0.6818 18 0.5769 23 0.5
1 0	0 1 2 3 4 5 6 7	10 6830 8.38861 e6 5.18282 e8 9.54437 e9 9.09091 e10 5.71545 e11 2.69971 e12	10 3.3502 2 2.0000 1 1.4285 7 1.1113 6 1.0314 8 6.2960 3 87.590 1	10 3.3585 5 2.0000 3 1.4285 7 1.1111 4 0.9405 88 2.6407 34.516 1	10 3.3723 2 2.0001 2 1.4285 7 1.1111 1 0.9142 96 1.2361 9 10.976 5	10 3.3939 7 2.0004 4 1.4285 7 1.1111 1 0.9095 91 0.8531 84 3.1220 4	10 3.4258 9 2.0013 7 1.4285 7 1.1111 1 0.9091 14 0.7797 96 1.1262 4	10 3.4548 6 2.0027 2 1.4285 7 1.1111 1 0.9090 92 0.7712 26 0.7923 19	10 3.5661 3 2.0127 3 1.4287 2 1.1111 1 0.9090 91 0.7692 36 0.6684 61	10 3.3333 3 2 1.4285 7 1.1111 1 0.9090 91 0.7692 31 0.6666 67

Table (1): Comparison between the 10th-order approximations of HPM, HAM and q-HAM at different values of n with the exact solution of (10).

4. CONCLUSION

An approximate solution of K(2,2) equation was found by using the q-homotopy analysis method (q-HAM).The results show that the convergence region of series solutions obtained by q-HAM is increasing as q is decreased. The comparison of q-HAM with the HAM and HPM was made. It was shown that the convergence of q-HAM is faster than the convergence of HAM and HPM.

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