FIXED POINT THEOREMS IN PARAMETRIC METRIC SPACE

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ABSTRACT

In this paper we proved the some fixed point theorems in Parametric metric spaces.

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INTRODUCTION AND PRELIMINARIES

Fixed point theorems are very important tool for proving the existence and eventually the uniqueness of the solutions to various mathematical models (integral and partial differ- ential equations, variational inequalities). In last few years different types of generalized metric spaces have been developed by different authors in different approach. Some generalized metric spaces are D-metric space, Cone metric space [4] etc. The notion of parametric metric spaces being a natural generalization of metric spaces was recently introduced and studied by Hussain et al. [2]. In this paper, we present some fixed point theorems under various expansive conditions in parametric metric spaces. These results improve and generalize some important known results in [6,7,8].

PRELIMINARIES

Definition 2.1. Let X be a nonempty set and a function

$$\rho : X \times X \times (0, +\infty) \rightarrow [0, +\infty)$$

is said to be a parametric metric on X if,

(1) $\rho(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$,

(2) $\rho(x, y, t) = \rho(y, x, t)$ for all $t > 0$,

(3) $\rho(x, y, t) \leq \rho(x, z, t) + \rho(z, y, t)$ for all $x, y, z \in X$ and all $t > 0$.

and the pair (X, ρ) is called parametric metric space.

Definition 2.2. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric metric space (X, ρ).

(1) $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to $x \in X$, if

$$\lim_{n \to \infty} \rho(x_n, x, t) = 0.$$

written as $\lim_{n \to \infty} x_n = x$, for all $t > 0$,

(2) $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence in X if for all $t > 0$, if

$$\lim_{n,m \to \infty} \rho(x_n, x_m, t) = 0.$$

(3) (X, ρ) is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 2.3. Let (X, ρ) be a parametric metric space and a function $T : X \to X$ is continuous at $x \in X$, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n \to \infty} x_n = x$, then $\lim_{n \to \infty} T x_n = T x$.

Example 2.4. Let $X = \{f \mid f : (0, +\infty) \to R\}$. And define the function $\rho : X \times X \times (0, +\infty) \to [0, +\infty)$ by $\rho(f, g, t) = |f(t) - g(t)|$, $\forall f, g \in X$ and all $t > 0$. Then ρ is a parametric metric on X and the pair (X, ρ) is a parametric metric space.

Lemma 2.5. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric metric space (X, ρ) such that

$$\rho(x_n, x_{n+1}, t) \le \lambda \rho(x_{n-1}, x_n, t)$$

where $\lambda \in [0, 1)$ and $n = 1, 2, \ldots$ Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, ρ).

Lemma 2.6. Let (X,ρ, s) be a parametric metric space with the coefficient $s = 1$. Let $\{x_n\}_{n=1}$ be a sequence of points of X such that

$$\rho(x_n, x_{n+1},t) \leq \lambda\rho(x_{n-1}, x_n,t)$$

where $\lambda \in [0, \frac{1}{s})$ and $n = 1, 2, \ldots$ Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X,ρ, s).

MAIN RESULTS

Theorem 3.1. Let (X, ρ) be a complete parametric metric space and T a continuous mapping satisfying the following condition:

$$\rho(Tx, Ty, t) \leq \alpha \max[\rho(x, y, t), \rho(x, T(x), t), \rho(y, T(y), t), \rho(x, T(y), t), \rho(T(x), y, t)]$$

for all $x, y \in X$, $x \neq y$, and for all $t > 0$, where $\alpha \in [0, 1)$. Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n\in N}$ as follows,

$Tx_n = x_{n+1}$ for $n = 1, 2, 3,\ldots$ Taking $x = x_n$ and $y = x_{n+1}$ in (1), we obtain

$$\rho(Tx_n,Tx_{n+1},t) \leq \alpha \max[\rho(x_n, x_{n+1}, t), \rho(x_n,Tx_n, t), \rho(x_{n+1},Tx_{n+1},t), \rho(x_n,Tx_{n+1},t), \rho(Tx_n, x_{n+1},t)].$$

$$\rho(x_{n+1}, x_{n+2},t) \leq \alpha \max[\rho(x_n, x_{n+1}, t), \rho(x_n, x_{n+1}, t), \rho(x_{n+1}, x_{n+2},t), \rho(x_n, x_{n+2}, t), \rho(x_{n+1}, x_{n+1},t)].$$

$$\rho(x_{n+1}, x_{n+2},t) \leq \alpha \max[\rho(x_n, x_{n+1}, t), \rho(x_n, x_{n+2},t)].$$

Case (i): If $\rho(x_{n+1}, x_{n+2},t) \leq \alpha\rho(x_n, x_{n+1},t)$. Hence by induction, we obtain

$$\rho(x_{n+1}, x_{n+2},t) \leq \alpha^{n+1}\rho(x_0, x_1,t), \qquad \forall t > 0 \quad \text{and} \quad \alpha < 1.$$

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By Lemma 2.5, $\{x_n\}_{n\in N}$ is a Cauchy sequence in X. But X is a complete parametric metric space; hence, $\{x_n\}_{n\in N}$ is converges. Call the limit $x^* \in X$. Then, $x_n \to x^*$ as $n \to +\infty$. By continuity of T we have,

$$Tx^* = T(\lim_{n\to\infty} x) = \lim_{n\to\infty} Tx = \lim_{n\to\infty} x_{n+1} = x^*.$$

That is, $Tx^* = x^*$; thus, T has a fixed point in X.

Case (ii): If $\rho(x_{n+1}, x_{n+2},t) \le \alpha\rho(x_n, x_{n+2},t)$.

$$\rho(x_{n+1}, x_{n+2},t) \le \alpha\,(\rho(x_n, x_{n+1},t) + \rho(x_{n+1}, x_{n+2},t))\,.$$

$$\le \frac{\alpha}{1-\alpha}\rho(x_n, x_{n+1},t).$$

$$\le h\rho(x_n, x_{n+1},t) \qquad \text{where} \quad h = \frac{\alpha}{1-\alpha} < 1$$

Hence by induction, we obtain

$$\rho(x_{n+1}, x_{n+2},t) \le h^{n+1}\rho(x_0, x_1,t)$$

By Lemma 2.5, $\{x_n\}_{n\in N}$ is a Cauchy sequence in X. But X is a complete parametric metric space; hence, $\{x_n\}_{n\in N}$ is converges. Call the limit $x^* \in X$. Then, $x_n \to x^*$ as $n \to +\infty$. By continuity of T we have,

$$Tx^* = T(\lim_{n\to\infty} x) = \lim_{n\to\infty} Tx = \lim_{n\to\infty} x_{n-1} = x^*.$$

That is, $Tx^* = x^*$; thus, T has a fixed point in X.

Uniqueness:

Let y^* be another fixed point of T in X; then $Ty^* = y^*$ and $Tx^* = x^*$. Now,

$$\rho(Tx^*, Ty^*, t) \le \alpha \max[\rho(x^*, y^*, t), \rho(x^*, Tx^*, t), \rho(y^*, Ty^*, t), \rho(x^*, Ty^*, t), \rho(Tx^*, y^*, t)].$$

This implies that

$$\rho(x^*, y^*, t) \le \alpha\rho(x^*, y^*, t)$$

This is true only when $\rho(x^*, y^*, t) = 0$. So $x^* = y^*$. Hence T has a unique fixed point in X.

□

Corollary 3.2. Let (X, ρ) be a complete parametric metric space and T a continuous mapping satisfying the following condition:

$$\rho(Tx, Ty, t) \le \alpha \max[\rho(x, y, t), \rho(x, T(x), t), \rho(y, T(y), t)]$$

for all $x, y \in X$, $x \neq y$, and $t > 0, \alpha \in [0, 1)$. Then T has a unique fixed point in X.

Proof. The proof of the corollary immediately follows since

$$\max[\rho(x, y, t), \rho(x, T(x), t), \rho(y, T(y), t)] \\ \le \max[\rho(x, y, t), \rho(x, T(x), t), \rho(y, T(y), t), \rho(x, T(y), t), \rho(T(x), y, t)]$$

□

Theorem 3.3. Let (X, ρ) be a complete parametric metric space and T a continuous mapping satisfying the following condition:

$$\rho(Tx, Ty, t) \le \alpha[\rho(x, T(x), t), \rho(y, T(y), t)] + \beta[\rho(x, T(y), t), \rho(T(x), y, t)]$$

for all $x, y \in X$, and $\alpha + \beta < \frac{1}{2}$, $\alpha, \beta \in \left[0, \frac{1}{2}\right)$. Then T has a fixed point in X.

Proof. Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n\in N}$ as follows, $Tx_n = x_{n+1}$ for $n = 1, 2, 3, \ldots$ Taking $x = x_n$ and $y = x_{n+1}$ in (1), we obtain

$$\begin{aligned}
\rho(Tx_n, Tx_{n+1}, t) &\le \alpha[\rho(x_n, T(x_n), t) + \rho(x_{n+1}, T(x_{n+1}), t)] \\
&\quad + \beta[\rho(x_n, T(x_{n+1}), t) + \rho(T(x_n), x_{n+1}, t)] \\
&\le \alpha[\rho(x_n, x_{n+1}, t) + \rho(x_{n+1}, x_{n+2}, t)] \\
&\quad + \beta[\rho(x_n, x_{n+2}, t) + \rho(x_{n+1}, x_{n+1}, t)] \\
&\le \alpha[\rho(x_n, x_{n+1}, t) + \rho(x_{n+1}, x_{n+2}, t)] \\
&\quad + \beta[\rho(x_n, x_{n+2}, t)] \\
\rho(x_{n+1}, x_{n+2}, t) &\le \alpha[\rho(x_n, x_{n+1}, t) + \rho(x_{n+1}, x_{n+2}, t)] \\
&\quad + \beta[\rho(x_{n+1}, x_n, t) + \rho(x_{n+1}, x_{n+2}, t)] \\
\rho(x_{n+1}, x_{n+2}, t) &\le (\alpha + \beta)[\rho(x_n, x_{n+1}, t) + \rho(x_{n+1}, x_{n+2}, t)]
\end{aligned}$$

$$\rho(x_{n+1}, x_{n+2}, t) \le L\rho(x_n, x_{n+1}, t) \quad \text{where } L = \frac{(\alpha + \beta)}{(1 - (\alpha + \beta))}$$

Hence by induction, we obtain

$$\rho(x_{n+1}, x_{n+2}, t) \leq L^{n+1}\rho(x_0, x_1, t)$$

By Lemma 2.5, $\{x_n\}_{n\in N}$ is a Cauchy sequence in X. But X is a complete parametric metric space; hence, $\{x_n\}_{n\in N}$ is converges. Call the limit $x^* \in X$. Then, $x_n \to x^*$ as $n \to +\infty$. By continuity of T we have,

$$Tx^* = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

That is, $Tx^* = x^*$. Thus, T has a fixed point in X. □

Theorem 3.4. Let (X, ρ) be a complete parametric metric space and T a continuous mapping satisfying the following condition:

$$\rho(Tx, Ty, t) + \alpha\rho(y, Tx, t) \geq \beta \frac{\rho(x, Tx, t)\rho(y, Ty, t)}{\rho(x, y, t)} + \gamma\rho(x, y, t)$$

for all $x, y \in X$, $x \neq y$, and for all $t > 0$, where $\alpha, \beta, \gamma \geq 0$ are real constants and $\beta + \gamma - 2\alpha > 1, \gamma - \alpha > 1$. Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n\in N}$ as follows,

$T x_n = x_{n-1}$ for $n = 1, 2, 3,...$ Taking $x = x_n$ and $y = x_{n+1}$ in (1), we obtain

$$\rho(Tx_{n+1}, Tx_{n+2}, t) + \alpha\rho(Tx_{n+2}, Tx_{n+1}, t) \geq \beta\frac{\rho(x_{n+1}, Tx_{n+1}, t)\rho(x_{n+2}, Tx_{n+2}, t)}{\rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t).$$

$$\rho(x_n, x_{n+1}, t) + \alpha\rho(x_{n+2}, x_n, t) \geq \beta\frac{\rho(x_{n+1}, x_n, t)\rho(x_{n+2}, x_{n+1}, t)}{\rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t).$$

$$\rho(x_n, x_{n+1}, t) + \alpha\rho(x_{n+2}, x_n, t) \geq \beta\frac{\rho(x_{n+1}, x_n, t)\rho(x_{n+1}, x_{n+2}, t)}{\rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t).$$

$$\rho(x_n, x_{n+1}, t) + \alpha\rho(x_n, x_{n+2}, t) \geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t)$$
$$\rho(x_n, x_{n+1}, t) + \alpha\rho(x_n, x_{n+1}, t) + \alpha\rho(x_{n+1}, x_{n+2}, t) \geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t)$$
$$(1 + \alpha - \beta)\rho(x_n, x_{n+1}, t) \geq (\gamma - \alpha)\rho(x_{n+1}, x_{n+2}, t)$$

for all $t > 0$. The last inequality gives

$$\rho(x_{n+1}, x_{n+2}, t) \leq \frac{1 + \alpha - \beta}{\gamma - \alpha}\rho(x_n, x_{n+1}, t)$$

$$= k\rho(x_n, x_{n+1}, t) \quad \text{where} \quad k = \frac{1 + \alpha - \beta}{\gamma - \alpha} < 1.$$

Hence by induction, we get

$$\rho(x_{n+1}, x_{n+2}, t) \leq k^{n+1}\rho(x_0, x_1, t)$$

By Lemma 2.5, $\{x_n\}_{n\in N}$ is a Cauchy sequence in X. But X is a complete parametric metric space; hence, $\{x_n\}_{n\in N}$ is converges. Call the limit $x^* \in X$. Then, $x_n \to x^*$ as $n \to +\infty$. By continuity of

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That is, $Tx^* = x^*$; thus, T has a fixed point in X.

Uniqueness:

Let y^* be another fixed point of T in X; then $Ty^* = y^*$ and $Tx^* = x^*$. Now,

$$\rho(Tx^*, Ty^*, t) \geq \alpha\rho(y^*, Tx^*, t) + \beta\frac{\rho(x^*, Tx^*, t)\rho(y^*, Ty^*, t)}{\rho(x^*, y^*, t)} + \gamma\rho(x^*, y^*, t)$$

$$\rho(x^*, y^*, t) + \alpha\rho(x^*, y^*, t) \geq \gamma\rho(x^*, y^*, t)$$

$$\rho(x^*, y^*, t) \geq (\gamma - \alpha)\rho(x^*, y^*, t)$$

$$\rho(x^*, y^*, t) \leq \frac{1}{\gamma - \alpha}\rho(x^*, y^*, t)$$

This is true only when $\rho(x^*, y^*, t) = 0$.

$$\Rightarrow \quad x^* = y^*.$$

Hence T has a unique fixed point in X. □

Theorem 3.5. Let (X, ρ) be a complete parametric metric space and T a continuous mapping satisfying the following condition:

$$\rho(Tx, Ty, t) + \alpha \min\{\rho(x, Ty, t), \rho(y, Tx, t)\} \geq \beta \frac{\rho(x, Tx, t)[\delta + \rho(y, Ty, t)]}{\delta + \rho(x, y, t)} + \gamma\rho(x, y, t)$$

for all $x, y \in X$, $x \neq y$, and for all $t > 0$, where $\delta \geq 1$, and $\alpha, \beta, \gamma \geq 0$ are real constants and $\beta + \gamma - \alpha > 1 + \alpha$, $\gamma > 1 + \alpha$. Then T has a fixed point in X.

Proof. Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n\in N}$ as follows, $T x_n = x_{n-1}$ for $n = 1, 2, 3,...$ Taking $x = x_n$ and $y = x_{n+1}$ in (1), we obtain

$$\rho(Tx_{n+1}, Tx_{n+2}, t) + \alpha \min\{\rho(x_{n+1}, Tx_{n+2}, t), \rho(x_{n+2}, Tx_{n+1} t)\} \geq \beta \frac{\rho(x_{n+1}, Tx_{n+1}, t)[\delta + \rho(x_{n+2}, Tx_{n+2}, t)]}{\delta + \rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t).$$

$$\rho(x_n, x_{n+1}, t) + \alpha \min\{\rho(x_{n+1}, x_{n+1}, t), \rho(x_{n+2}, x_n, t)\} \geq \beta \frac{\rho(x_{n+1}, x_n, t)[\delta + \rho(x_{n+2}, x_{n+1}, t)]}{\delta + \rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t).$$

$$\rho(x_n, x_{n+1}, t) + \alpha \min\{\rho(x_{n+1}, x_{n+1}, t), \rho(x_{n+2}, x_n, t)\} \geq \beta \frac{\rho(x_{n+1}, x_n, t)[\delta + \rho(x_{n+1}, x_{n+2}, t)]}{\delta + \rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t).$$

$$\rho(x_n, x_{n+1}, t) + \alpha\rho(x_n, x_{n+2}, t) \geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t)$$

$$\rho(x_n, x_{n+1}, t) + \alpha\rho(x_n, x_{n+1}, t) + \alpha\rho(x_{n+1}, x_{n+2}, t) \geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t)$$

$$(1 + \alpha - \beta)\rho(x_n, x_{n+1}, t) \geq (\gamma - \alpha)\rho(x_{n+1}, x_{n+2}, t)$$

for all $t > 0$. The last inequality gives

$$\rho(x_{n+1}, x_{n+2}, t) \leq \frac{1+\alpha-\beta}{\gamma-\alpha}\rho(x_n, x_{n+1}, t) = k\rho(x_n, x_{n+1}, t)$$

where $k = \frac{1+\alpha-\beta}{\gamma-\alpha} < 1$. Hence by induction, we obtain

$$\rho(x_{n+1}, x_{n+2}, t) \leq k^{n+1}\rho(x_0, x_1, t)$$

By Lemma 2.5, $\{x_n\}_{n\in N}$ is a Cauchy sequence in X. But X is a complete parametric metric space; hence, $\{x_n\}_{n\in N}$ is converges. Call the limit $x^* \in X$. Then, $x_n \to x^*$ as $n \to +\infty$. By continuity of T we have,

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That is, $Tx^* = x^*$; thus, T has a fixed point in X.

Uniqueness:

Let y^* be another fixed point of T in X; then $Ty^* = y^*$ and $Tx^* = x^*$. Now,

$$\rho(Tx^*, Ty^*, t) + \alpha \min\{\rho(x^*, Ty^*, t), \rho(y^*, Tx^*, t)\} \geq \beta \frac{\rho(x^*, Tx^*, t)[\delta + \rho(y^*, Ty^*, t)]}{\delta + \rho(x^*, y^*, t)}$$

$$+ \gamma\rho(x^*, y^*, t) \quad \rho(x^*, y^*, t) + \alpha\rho(x^*, y^*, t) \geq \gamma\rho(x^*, y^*, t)$$

$$\rho(x^*, y^*, t) \geq (\gamma - \alpha)\rho(x^*, y^*, t)$$

$$\rho(x^*, y^*, t) \leq \frac{1}{\gamma - \alpha}\rho(x^*, y^*, t)$$

This is true only when $\rho(x^*, y^*, t) = 0$.

$$\Rightarrow \quad x^* = y^*.$$

Hence T has a unique fixed point in X. □

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